STOCHASTIC PROCESSES

UNIT - I :

Definition of stochastic processes - classification of stochastic processes according to time parameter space and state space - Example of stochastic processes.

UNIT - II :

Markov chains - Definitions and examples - Higher transition probabilities - Chapman-Kolmogorov equation - classification of states - Limiting behaviour (concept and applications only)

UNIT - III :

Stationary processes and time series - Strict and wide sense stationary models of time series - concept of spectrum of time series.

UNIT - IV :

Poisson processes - Poisson process and related distributions - Birth - death processes - Simple examples.

UNIT - V :

Markov process with continuous state space - Brownian movement - Wiener process -

Differential equation for a wiener process - kolmogorov equations - first passage time distribution for wienor process - Distribution of the maximum of a wiener process - Distribution of the first passage time to a fixed point.

REFERENCE BOOKS :

1. S. Karlin, H. M. Taylor (1966) : First course in stochastic processes, Academic press.

2. J. Medhi (1982) : Stochastic processes, Wiley Eastern Ltd., New Delhi

3. N.U. Prabhu (1955) : Stochastic processes, Mac. Millan, New york.

UNIT - I

Introduction :

Due to the changes in the approach to the scientific enquiries, stochastic models are more realistic than the deterministic model in many situations. Observations taken from different time point attracts the attention of a probabilists than the observations are taken at fixed time points.

Example :

The moment of population expectation of life, labour force, national income etc can be considered a time dependent random variables.

The concept of dynamic indeterminism is used to study the changing pattern of the above set of variables with respect to time or space.

A sequence of random variables doesnot specify the dynamic structure of the problem but the collection of random variables specify the dynamic structure of the problem.

Example:

The no. of transition made by radio active substances in a given time interval $(0, t)$ is a poisson random variable with suitable parameters.

i.e, The pmf of poisson random variable x is,

$$P\{x = x, \text{ given time } 't'\} = \frac{e^{-\lambda} \lambda^{x}}{x!} \; ; \; x = 0, 1, 2, \ldots$$

It gives a quick comparison with the phenomenon for different time point. Further, it shows that the parameter λ is a function of 't'. Further another question arises about the distribution of time interval between successive radio active transition. In such situations, sequence of random variable is not appropriate one to carry out the analysis. To meet the deficiency, a general collection of r.v rather than the sequence of r.v is used to describe the dynamic structure of the problem and to make the analysis relevant to the real situations.

Stochastic Process - Def :

A general collection of r.v is called as a random process (or) stochastic process. It may be defined as follows.

A stochastic process is a family of r.v $\{x(t), t \in T\}$ defined on a common probability space $\{\Omega, A, P\}$ where T is a subset of real line and it is called as parametric set (or) Index set of the process.

A stochastic process is defined as a set of functions $\{x(\omega), \omega \in \Omega\}$ on a parametric space T.

Thus the stochastic process is either a collection of r.v $x(t)$, depending on the time parameter 't' (or) as a collection of the set function $\{x(\omega), \omega \in \Omega\}$.

The stochastic process is continuous parametric process, if 'T' is an interval having (+)ve length. If T is a subset of integers then the S.P is called discrete parametric process.

State Space

The set of all possible values taken by the r.v's in the process $\{x(t), t \in T\}$ is called as state space.

Eg:

Let $\{l_x\}$ be the life table function representing the no. of survivors at each age x; $x = 0, 1, 2, \ldots$ then the collection of r.v's $\{l_x, x = 0, 1, 2, \ldots\}$ is called a S.P where x is the index parameter and it belongs to the discrete parametric space.

The set of values of l_x is called as the state space of the process.

Classification of Stochastic Process:

Stochastic process may be classified with respect to the state space and parametric space.

Let $\{X(t), t \in T\}$ be a stochastic process with parametric space T and state space x. It may be classified as

(i) Discrete value S.P when both x & T are discrete

(ii) Discrete value continuous parameter S.P if x is discrete and T is continuous

(iii) continuous S.P with a discrete parameter, if x is continuous and T is discrete

(iv) continuous S.P if both x & T are continuous.

case (i) :

A S.P may have x (state space) and T (parametric space) as discrete

Eg :

1. The no. of bread slices in a stock on different days. The no. of bread leaves are represented by the r.v X and the no. of days are represented by the parametric space T.

Here $\{X(t), t \in T\}$ is a S.P with both spaces are discrete.

2. The weather condition at 2 PM on different days are also as example for a S.P with discrete state space and discrete parametric space.

case (ii) :

A S.P may have x (state space) as discrete and T (parametric space) as continuous.

Eg :

1. The no. of persons waiting for bus in a bus stand at any time of the day.

Here the collection $\{X(t), t \in T\}$ is a discrete value S.P with continuous parametric space, where $X(t)$ represents the no. of persons waiting for a bus in a busstand at time t. Here the state space $x = \{0, 1, \dots n\}$, the parametric space $T = \{t / 0 < t < T (= 24)\}$

2. The no. of accidents at any time in a day is an example for a S.P with discrete state space and continuous parametric space.

case (iii) :

A S.P may have a continuous state space and discrete parametric space

Eg :

The stock level of wheat in warehouse at the beginning of every month is an example for a S.P with continuous state space and discrete parametric space.

case (iv) :

A S.P may have continuous state space and continuous parametric space

Here $\{X(t), t \in T\}$ is a S.P with continuous state space and continuous parametric space, where $X(t)$ represents the contents (water level) of a dam at time 't'.

A S.P with stationary independent increments

If for all $t_1, t_2, t_3, \ldots, t_n$ such that $t_1 < t_2 < \ldots < t_n$ then the r.v's $x(t_2) - x(t_1)$, $x(t_3) - x(t_2)$, $\ldots$, $x(t_n) - x(t_{n-1})$ are all independent then the S.P $\{x(t), t \in T\}$ is a stationary S.P with independent increments. Because all the distribution fn. of $x(t_k) - x(t_{k-1})$ $\forall$ $k = 1, 2, \ldots n$ remains invariant for the location transformation (shifting of the origin for each k).

Stationary process:

A S.P is said to be stationary if its finite dimensional distributions are invariant under arbitrary translation of the time parameter. ie The process $\{x(t), t \in T\}$ $\forall n$ is stationary if for all n.

(or)

$x(t_1 + h) - x(t_1)$, the increment purely depends on 'h' not on 't_1' time points. The time points may be start time point (or) End time point. This process is known as stationary process.

Strict stationary :

$$P\left\{x(t_1) \le x_1, x(t_2) \le x_2, \dots x(t_n) \le x_n\right\} =$$

$$P\left\{x(t_1+h) \le x_1, x(t_2+h) \le x_2, \dots x(t_n+h) \le x_n\right\}$$

may also called as strict stationary process $\forall\, t_i \in T, \forall\, t_i + h \in T, \forall\, i = 1, 2, \dots k, h > 0$

wide sense stationary :

The S.P is called as covariance stationary (or) wide sense stationary (or) weekly stationary process if the covariance between $x(t)$ and $x(t+h)$

ie $\operatorname{cov}[x(t), x(t+h)] = E[x(t).x(t+h)] - E[x(t)] \cdot E[x(t+h)]$

depends on 'h' for $t \in T$.

Example :

1. Let us consider the S.P $\{x_n, n \ge 1\}$ when x_n be uncorrelated r.v's with mean 0 and variance 1.

The covariance between $x(t)$, $x(t+h)$ is such as

$$\operatorname{cov}[x(t), x(t+h)] = E[x(t).x(t+h)] - E[x(t)] \cdot E[x(t+h)]$$

$$= E[X(t) \cdot X(t+h)] - 0$$

$$\left\{ \because E[X(t)] \cdot E[X(t+h)] = 0 \right\}$$

$$= E[X(t)] \cdot E[X(t+h)]$$

$$= \begin{cases} 0 & \text{if } h > 0 \\ 1 & \text{if } h = 0 \end{cases}$$

Thus the process $\{X_n, n \geq 1\}$ is a covariance stationary because the covariance between $X(t)$ & $X(t+h)$ depends on h.

2. Let us consider the S.P $\{X(t), t \in T\}$, where $X(t)$ is a r.v with pmf

$$P[X(t) = x] = \frac{e^{-at}(at)^x}{x!} \quad \forall\ x = 0, 1, 2, \ldots \ \& \ a > 0$$

Mean and variance is given by

$$E[X(t)] = at$$

$$V[X(t)] = at$$

Since the mean & variance of $X(t)$ depends on the parameter T and it implies the distribution function depends on the parameter T. Hence the S.P is not stationary but it is said to be "Evolutionary".